MASAGA: A Linearly-Convergent Stochastic First-Order Method for Optimization on Manifolds

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Abstract. We consider the stochastic optimization of finite sums over a Riemannian manifold where the functions are smooth and convex. We present MASAGA, an extension of the stochastic average gradient variant SAGA on Riemannian manifolds. SAGA is a variance-reduction technique that typically outperforms methods that rely on expensive full-gradient calculations, such as the stochastic variance-reduced gradient method. We show that MASAGA achieves a linear convergence rate with uniform sampling, and we further show that MASAGA achieves a faster convergence rate with non-uniform sampling. Our experiments show that MASAGA is faster than the recent Riemannian stochastic gradient descent algorithm for the classic problem of finding the leading eigenvector corresponding to the maximum eigenvalue.

Keywords: Variance Reduced Stochastic Optimization· Riemannian Manifold.

1 Introduction

The most common supervised learning methods in machine learning use empirical risk minimization during the training. The minimization problem can be expressed as minimizing a finite sum of loss functions that are evaluated at a single data sample. We consider the problem of minimizing a finite sum over a Riemannian manifold,

$$\min_{x \in \mathcal{X} \subseteq \mathcal{M}} f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

where $\mathcal{X}$ is a geodesically convex set in the Riemannian manifold $\mathcal{M}$. Each function f_i is geodesically Lipschitz-smooth and the sum is geodesically strongly-convex over the set $\mathcal{X}$. The learning phase of several machine learning models can be written as an optimization problem of this form. This includes principal component analysis (PCA) [39], dictionary learning [34], Gaussian mixture models (GMM) [10], covariance estimation [36], computing the Riemannian centroid [11], and PageRank algorithm [33].

When $\mathcal{M} \equiv \mathbb{R}^d$, the problem reduces to convex optimization over a standard Euclidean space. An extensive body of literature studies this problem in deterministic and stochastic settings [5,29,22,28,23]. It is possible to convert the optimization over a manifold into an optimization in a Euclidean space by adding $x \in \mathcal{X}$ as an optimization constraint. The problem can then be solved using projected-gradient methods. However, the problem with this approach is that we are not explicitly exploiting the geometrical structure of the manifold. Furthermore, the projection step for the most common non-trivial manifolds used in practice (such as the space of positive-definite matrices) can be quite expensive. Further, a function could be non-convex in the Euclidean space but geodesically convex over an appropriate manifold. These factors can lead to poor performance for algorithms that operate with the Euclidean geometry, but algorithms that use the Riemannian geometry may converge as fast as algorithms for convex optimization in Euclidean spaces.

Stochastic optimization over manifolds and their convergence properties have received significant interest in the recent literature [4,38,14,30,37]. Bonnabel [4] and Zhang *et al.* [38] analyze the application of stochastic gradient descent (SGD) for optimization over manifolds. Similar to optimization over Euclidean spaces with SGD, these methods suffer from the aggregating variance problem [40] which leads to sublinear convergence rates.

When optimizing finite sums over Euclidean spaces, variance-reduction techniques have been introduced to reduce the variance in SGD in order to achieve faster convergence rates. The variance-reduction techniques can be categorized into two groups. The first group is memory-based approaches [16,32,20,6] such as the stochastic average gradient (SAG) method and its variant SAGA. Memory-based methods use the memory to store a stale gradient of each f_i, and in each iteration they update this “memory” of the gradient of a random f_i. The averaged stored value is used as an approximation of the gradient of f.

The second group of variance-reduction methods explored for Euclidean spaces require full gradient calculations and include the stochastic variance-reduced gradient (SVRG) method [12] and its variants [19,15,25]. These methods only store the gradient of f, and not the gradient of the individual f_i functions. But, these methods occasionally require evaluating the full gradient of f as part of their gradient approximation and require two gradient evaluations per iteration. Although SVRG often dramatically outperforms the classical gradient descent (GD) and SGD, the extra gradient evaluation typically lead to a slower convergence than memory-based methods. Furthermore, the extra gradient calculations of SVRG can lead to worse performance than the classical SGD during the early iterations where SGD has the most advantage [9]. Thus, when the bottleneck of the process is the gradient computation itself, using memory-based methods like SAGA can improve performance [7,3]. Furthermore, for several applications it has been shown that the memory requirements can be alleviated by exploiting special structures in the gradients of the f_i [16,32,31].

Several recent methods have extended SVRG to optimize the finite sum problem over a Riemannian manifold [14,30,37], which we refer to as RSVRG methods. Similar to the case of Euclidean spaces, RSVRG converges linearly for geodesically Lipschitz-smooth and strongly-convex functions. However, these methods also require the extra gradient evaluations associated with the original SVRG method. Thus, they may not perform as well as potential generalizations of memory-based methods like SAGA.

In this work we present *MASAGA*, a variant of SAGA to optimize finite sums over Riemannian manifolds. Similar to RSVRG, we show that it converges linearly for geodesically strongly-convex functions. We also show that both MASAGA and

RSVRG with a non-uniform sampling strategy can converge faster than the uniform sampling scheme used in prior work. Finally, we consider the problem of finding the leading eigenvector, which minimizes a quadratic function over a sphere. We show that MASAGA converges linearly with uniform and non-uniform sampling schemes on this problem. For evaluation, we consider one synthetic and two real datasets. The real datasets are MNIST [17] and the Ocean data [18]. We find the leading eigenvector of each class and visualize the results. On MNIST, the leading eigenvectors resemble the images of each digit class, while for the Ocean dataset we observe that the leading eigenvector represents the background image in the dataset.

In Section 2 we present an overview of essential concepts in Riemannian geometry, defining the geodesically convex and smooth function classes following Zhang *et al.* [38]. We also briefly review the original SAGA algorithm. In Section 3, we introduce the MASAGA algorithm and analyze its convergence under both uniform and non-uniform sampling. Finally, in Section 4 we empirically verify the theoretical linear convergence results.

2 Preliminaries

In this section we first present a review of Riemannian manifold concepts, however, for a more detailed review we refer the interested reader to the literature [27,1,35]. Then, we introduce the class of functions that we optimize over such manifolds. Finally, we briefly review the original SAGA algorithm.

2.1 Riemannian Manifold

A Riemannian manifold is denoted by the pair $(\mathcal{M}, G)$, that consists of a smooth manifold $\mathcal{M}$ over $\mathbb{R}^d$ and a metric G. At any point x in the manifold $\mathcal{M}$, we define $\mathcal{T}_{\mathcal{M}}(x)$ to be the tangent plane of that point, and G defines an inner product in this plane. Formally, if p and q are two vectors in $\mathcal{T}_{\mathcal{M}}(x)$, then $\langle p, q \rangle_x = G(p, q)$. Similar to Euclidean space, we can define the norm of a vector and the angle between two vectors using G.

To measure the distance between two points on the manifold, we use the geodesic distance. Geodesics on the manifold generalize the concept of straight lines in Euclidean space. Let us denote a geodesic with $\gamma(t)$ which maps $[0,1] \to \mathcal{M}$ and is a function with constant gradient,

$$\frac{d^2}{dt^2}\gamma(t) = 0.$$

To map a point in $\mathcal{T}_{\mathcal{M}}(x)$ to $\mathcal{M}$, we use the exponential function $\mathrm{Exp}_x : \mathcal{T}_{\mathcal{M}}(x) \to \mathcal{M}$. Specifically, $\mathrm{Exp}_x(p) = z$ means that there is a geodesic curve $\gamma_x^z(t)$ on the manifold that starts from x (so $\gamma_x^z(0) = x$) and ends at z (so $\gamma_x^z(1) = z = \mathrm{Exp}_x(p)$) with a velocity of p ($\frac{d}{dt}\gamma_x^z(0) = p$). When the Exp function is defined for every point in the manifold, we call the manifold geodesically-complete. For example, the unit sphere in $\mathcal{R}^n$ is geodesically complete. If there is a unique geodesic curve between any two points in $\mathcal{M}' \in \mathcal{M}$, then the Exp_x function has an inverse defined by the Log_x function. Formally the $\mathrm{Log}_x \equiv \mathrm{Exp}_x^{-1} : \mathcal{M}' \to \mathcal{T}_{\mathcal{M}}(x)$ function maps a point from $\mathcal{M}'$ back into the tangent plane at x. Moreover, the geodesic distance between x and z is the length of the unique shortest path between z and x, which is equal to $\|\mathrm{Log}_x(z)\| = \|\mathrm{Log}_z(x)\|$.

Let u and $v \in \mathcal{T}_{\mathcal{M}}(x)$ be linearly independent so they specify a two dimensional subspace $\mathcal{S}_x \in \mathcal{T}_{\mathcal{M}}(x)$. The exponential map of this subspace, $\mathrm{Exp}_x(\mathcal{S}_x) = \mathcal{S}_{\mathcal{M}}$, is a two dimensional submanifold in $\mathcal{M}$. The sectional curvature of S_M denoted by $\mathrm{K}(\mathcal{S}_{\mathcal{M}}, x)$ is defined as a Gauss curvature of S_M at x [41]. This sectional curvature helps us in the convergence analysis of the optimization method. We use the following lemma in our analysis to give a trigonometric distance bound.

Lemma 1. *(Lemma 5 in [38]) Let a, b, and c be the side lengths of a geodesic triangle in a manifold with sectional curvature lower-bounded by $K_{\min}$. Then*

$$a^2 \leq \frac{c\sqrt{|K_{\min}|}}{\tanh(\sqrt{|K_{\min}|}c)} b^2 + c^2 - 2bc\cos(\measuredangle(b,c)).$$

Another important map used in our algorithm is the parallel transport. It transfers a vector from a tangent plane to another tangent plane along a geodesic. This map is denoted by $\Gamma_x^z : \mathcal{T}_{\mathcal{M}}(x) \to \mathcal{T}_{\mathcal{M}}(z)$, and maps a vector from the tangent plane $\mathcal{T}_{\mathcal{M}}(x)$ to a vector in the tangent plane $\mathcal{T}_{\mathcal{M}}(z)$ while preserving the norm and inner product values.

$$\langle p, q \rangle_x = \langle \Gamma_x^z(p), \Gamma_x^z(q) \rangle_z$$

Grassmann manifold. Here we review the Grassmann manifold, denoted $\mathrm{Grass}(p,n)$, as a practical Riemannian manifold used in machine learning. Let p and n be positive integers with $p \leq n$. $\mathrm{Grass}(p,n)$ contains all matrices in $\mathbb{R}^{n\times p}$ with orthonormal columns (the class of orthogonal matrices). By the definition of an orthogonal matrix, if $M \in \mathrm{Grass}(p,n)$ then we have $M^\top M = I$, where $I \in \mathbb{R}^{p\times p}$ is the identity matrix. Let $q \in \mathcal{T}_{\mathrm{Grass}(p,n)}(x)$, and $q = U\Sigma V^\top$ be its p-rank singular value decomposition. Then we have:

$$\mathrm{Exp}_x(tq) = xV\cos(t\Sigma)V^\top + U\sin(t\Sigma)V^\top.$$

The parallel transport along a geodesic curve $\gamma(t)$ such that $\gamma(0) = x$ and $\gamma(1) = z$ is defined as:

$$\Gamma_x^z(tq) = (-xV\sin(t\Sigma)U^\top + U\cos(t\Sigma)U^\top + I - UU^\top)q.$$

2.2 Smoothness and Convexity on Manifold

In this section, we define convexity and smoothness of a function over a manifold following Zhang *et al.* [38]. We call $\mathcal{X} \in \mathcal{M}$ geodesically convex if for any two points y and z in $\mathcal{X}$, there is a geodesic $\gamma(t)$ starting from y and ending in z with a curve inside of $\mathcal{X}$. For simplicity, we drop the subscript in the inner product notation.

Formally, a function $f : \mathcal{X} \to \mathbb{R}$ is called geodesically convex if for any y and z in $\mathcal{X}$ and the corresponding geodesic γ, for any $t \in [0,1]$ we have:

$$f(\gamma(t)) \leq (1-t)f(y) + tf(z).$$

Algorithm 1 The Original SAGA Algorithm

```
1: **Input:** Learning rate $\eta$.
2: Initialize $x_0 = 0$ and memory $M^{(0)}$ with gradient of $x_0$.
3: **for** $t = 1, 2, 3, \ldots$ **do**
4:     $\hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} M^t[j]$
5:     Pick $i_t$ uniformly at random from $\{1 \ldots n\}$.
6:     $\nu_t = \nabla f_{i_t}(x_t) - M^t[i_t] + \frac{1}{n}\sum_{j=1}^{n} M^t[j]$
7:     $x_{t+1} = x_t - \eta(\nu_t)$
8:     Set $M^{t+1}[i_t] = \nabla f_{i_t}(x_t)$ and $M^{t+1}[j] = M^t[j]$ for all $j \neq i_t$.
9: **end for**
```

Similar to the Euclidean space, if the Log function is well defined we have the following for convex functions:

$$f(z) + \langle g_z, \operatorname{Log}_z(y)\rangle \leq f(y),$$

where g_z is a subgradient of f at x. If f is a differentiable function, the Riemannian gradient of f at z is a vector g_z which satisfies $\frac{d}{dt}|_{t=0} f(\operatorname{Exp}_z(tg_z)) = \langle g_z, \nabla f(z)\rangle_z$, with ∇f being the gradient of f in $\mathbb{R}^n$. Furthermore, we say that f is geodesically μ-strongly convex if there is a $\mu > 0$ such that:

$$f(z) + \langle g_z, \operatorname{Log}_z(y)\rangle + \frac{\mu}{2}\|\operatorname{Log}_z(y)\|^2 \leq f(y).$$

Let $x^* \in \mathcal{X}$ be the optimum of f. This implies that there exists a subgradient at x^* with $g_{x^*} = 0$ which implies that the following inequalities hold:

$$\|\operatorname{Log}_z(x^*)\|^2 \leq \frac{2}{\mu}(f(z) - f(x^*))$$

$$\langle g_z, \operatorname{Log}_z(x^*)\rangle + \frac{\mu}{2}\|\operatorname{Log}_z(x^*)\|^2 \leq 0$$

Finally, an f that is differentiable over $\mathcal{M}$ is said to be a Lipschitz-smooth function with the parameter $L > 0$ if its gradient satisfies the following inequality:

$$\|g_z - \Gamma_y^z\left[g_y\right]\| \leq L\|\operatorname{Log}_z(y)\| = L\,\mathrm{d}(z, y),$$

where $\mathrm{d}(z, y)$ is the distance between z and y. For a geodesically smooth f the following inequality also holds:

$$f(y) \leq f(z) + \langle g_z, \operatorname{Log}_z(y)\rangle + \frac{L}{2}\|\operatorname{Log}_z(y)\|^2.$$

2.3 SAGA Algorithm

In this section we briefly review the SAGA method [6] and the assumptions associated with it. SAGA assumes f is μ-strongly convex, each f_i is convex, and each gradient ∇f_i is Lipschitz-continuous with constant L. The method generates a sequence of iterates x_t using the SAGA Algorithm 1 (line 7). In the algorithm, M is the memory used to store stale gradients. During each iteration, SAGA picks one f_{i_t} randomly and evaluates its gradient at the current iterate value, $\nabla f_{i_t}(x_t)$. Next, it computes ν_t as the difference between the current $\nabla f_{i_t}(x_t)$ and the corresponding stale gradient of f_{i_t} stored in the memory plus the average of all stale gradients (line 6). Then it uses this vector ν_t as an approximation of the full gradient and updates the current iterate similar to the gradient descent update rule. Finally, SAGA updates the stored gradient of f_{i_t} in the memory with the new value of $\nabla f_{i_t}(x_t)$.

Let $\rho_{\text{saga}} = \frac{\mu}{2(n\mu+L)}$. Defazio *et al.* [6] show that the iterate value x_t converges to the optimum x^* linearly with a contraction rate $1 - \rho_{\text{saga}}$,

$$\mathbb{E}\left[\|x_t - x^*\|^2\right] \leq (1 - \rho_{\text{saga}})^t C,$$

where C is a positive scalar.

3 Optimization on Manifold with SAGA

In this section we introduce the MASAGA algorithm (see Alg. 2). We make the following assumptions:

1. Each f_i is geodesically L-Lipschitz continuous.
2. f is geodesically μ-strongly convex.
3. f has an optimum in $\mathcal{X}$, *i.e.*, $x^* \in \mathcal{X}$.
4. The diameter of $\mathcal{X}$ is bounded above, *i.e.*, $\max_{u,v \in \mathcal{X}} \mathrm{d}(u,v) \leq D$.
5. Log_x is defined when $x \in \mathcal{X}$.
6. The sectional curvature of $\mathcal{X}$ is bounded, *i.e.*, $K_{\min} \leq K_{\mathcal{X}} \leq K_{\max}$.

These assumptions also commonly appear in the previous work [38,14,30,37]. Similar to the previous work [38,14,37], we also define the constant ζ which is essential in our analysis:

$$\zeta = \begin{cases} \frac{\sqrt{|K_{\min}|}D}{\tanh(\sqrt{|K_{\min}|}D)} & \text{if } K_{\min} < 0 \\ 1 & \text{if } K_{\min} \geq 0 \end{cases}$$

In MASAGA we modify two parts of the original SAGA: (i) since gradients are in different tangent planes, we use parallel transport to map them into the same tangent plane and then do the variance reduction step (line 6 of Alg. 2), and (ii) we use the Exp function to map the update step back into the manifold (line 7 of Alg. 2).

3.1 Convergence Analysis

We analyze the convergence of MASAGA considering the above assumptions and show that it converges linearly. In our analysis, we use the fact that MASAGA's estimation of the full gradient ν_t is unbiased (like SAGA), *i.e.*, $\mathbb{E}[\nu_t] = \nabla f(x_t)$. For simplicity, we use ∇f to denote the Riemannian gradient instead of g_x. We assume that there exists an incremental first-order oracle (IFO) [2] that gets an $i \in \{1, ..., n\}$, and an $x \in \mathcal{X}$, and return $(f_i(x), \nabla f_i(x)) \in (\mathbb{R} \times \mathcal{T}_{\mathcal{M}}(x))$.

Algorithm 2 MASAGA Algorithm

```
Input: Learning rate $\eta$ and $x_0 \in \mathcal{M}$.
Initialize memory $M^{(0)}$ with gradient of $x_0$.
for $t = 1, 2, 3, \dots$ do
    $\hat{\mu} = \frac{1}{n} \sum_{j=1}^{n} M^t[j]$
    Pick $i_t$ uniformly at random from $\{1 \dots n\}$.
    $\nu_t = \nabla f_{i_t}(x_t) - \varGamma_{x_0}^{x_t}\left[M^t[i_t] - \hat{\mu}\right]$
    $x_{t+1} = \mathrm{Exp}_{x_t}(-\eta(\nu_t))$
    Set $M^{t+1}[i_t] = \varGamma_{x_t}^{x_0}\left[\nabla f_{i_t}(x_t)\right]$ and $M^{t+1}[j] = M^t[j]$ for all $j \neq i_t$.
end for
```

Theorem 1. *If each f_i is geodesically L-smooth and f is geodesically μ-strongly convex over the Riemannian manifold $\mathcal{M}$, the MASAGA algorithm with the constant step size $\eta = \frac{2\mu + \sqrt{\mu^2 - 8\rho(1+\alpha)\zeta L^2}}{4(1+\alpha)\zeta L^2}$ converges linearly while satisfying the following:*

$$\mathbb{E}\left[\mathrm{d}^2(x_t, x^*)\right] \leq (1-\rho)^t \varUpsilon^0,$$

where $\rho = \min\{\frac{\mu^2}{8(1+\alpha)\zeta L^2}, \frac{1}{n} - \frac{1}{\alpha n}\}$, $\varUpsilon^0 = 2\alpha\zeta\eta^2 \sum_{i=1}^{n} \|M^0[i] - \varGamma_{x^*}^{x_0}\left[\nabla f_i(x^*)\right]\|^2 + \mathrm{d}^2(x_0, x^*)$ is a positive scalar, and $\alpha > 1$ is a constant.

Proof. Let $\delta_t = \mathrm{d}^2(x_t, x^*)$. First we find an upper-bound for $\mathbb{E}\left[\|\nu_t\|^2\right]$.

$$\begin{aligned}
\mathbb{E}\left[\|\nu_t\|^2\right] &= \mathbb{E}\left[\|\nabla f_{i_t}(x_t) - \varGamma_{x_0}^{x_t}\left[M^t[i_t] - \hat{\mu}\right]\|^2\right] \\
&= \mathbb{E}\left[\|\nabla f_{i_t}(x_t) - \varGamma_{x^*}^{x_t}\left[\nabla f_{i_t}(x^*)\right] - \varGamma_{x_0}^{x_t}\left[M^t[i_t] - \varGamma_{x^*}^{x_0}\left[\nabla f_{i_t}(x^*)\right] - \hat{\mu}\right]\|^2\right] \\
&\leq 2\mathbb{E}\left[\|\nabla f_{i_t}(x_t) - \varGamma_{x^*}^{x_t}\left[\nabla f_{i_t}(x^*)\right]\|^2\right] \\
&+ 2\mathbb{E}\left[\|\varGamma_{x_0}^{x_t}\left[M^t[i_t] - \varGamma_{x^*}^{x_0}\left[\nabla f_{i_t}(x^*)\right] - \hat{\mu}\right]\|^2\right] \\
&\leq 2\mathbb{E}\left[\|\nabla f_{i_t}(x_t) - \varGamma_{x^*}^{x_t}\left[\nabla f_{i_t}(x^*)\right]\|^2\right] \\
&+ 2\mathbb{E}\left[\|M^t[i_t] - \varGamma_{x^*}^{x_0}\left[\nabla f_{i_t}(x^*)\right]\|^2\right] \\
&\leq 2L^2\delta_t + 2\mathbb{E}\left[\|M^t[i_t] - \varGamma_{x^*}^{x_0}\left[\nabla f_{i_t}(x^*)\right]\|^2\right]
\end{aligned}$$

The first inequality is due to $(a+b)^2 \leq 2a^2 + 2b^2$ and the second one is from the variance upper-bound inequality, *i.e.*, $\mathbb{E}\left[x^2 - \mathbb{E}\left[x\right]^2\right] \leq \mathbb{E}\left[x^2\right]$. The last inequality comes from the geodesic Lipschitz smoothness of each f_i. Note that the expectation is taken with respect to i_t.

$$\begin{aligned}
\mathbb{E}\left[\delta_{t+1}\right] &\leq \mathbb{E}\left[\delta_t - 2\left\langle \nu_t, \mathrm{Exp}_{x_t}^{-1}(-x^*)\right\rangle + \zeta\eta^2\|\nu_t\|^2\right] \\
&= \delta_t - 2\eta\left\langle \nabla f(x_t), \mathrm{Exp}_{x_t}^{-1}(-x^*)\right\rangle + \zeta\eta^2\mathbb{E}\left[\|\nu_t\|^2\right] \\
&\leq \delta_t - \eta\mu\delta_t + \zeta\eta^2\mathbb{E}\left[\|\nu_t\|^2\right] \\
&\leq (1-\mu\eta)\delta_t + \zeta\eta^2\left[2L^2\delta_t + 2\mathbb{E}\left[\|M^t[i_t] - \varGamma_{x^*}^{x_0}\left[\nabla f_{i_t}(x^*)\right]\|^2\right]\right] \\
&= (1 - \mu\eta + 2\zeta L^2\eta^2)\delta_t + 2\zeta\eta^2\varPsi_t
\end{aligned}$$

The first inequality is due to the trigonometric distance bound, the second one is due to the strong convexity of f, and the last one is due to the upper-bound of ν_t. Ψ_t is defined as follows:

$$\Psi_t = \frac{1}{n}\sum_{i=1}^{n} \|M^t[i] - \Gamma_{x^*}^{x_0}\left[\nabla f_i(x^*)\right]\|^2.$$

We define the Lyaponov function

$$\Upsilon^t = \delta_t + c\Psi_t$$

for some $c > 0$. Note that $\Upsilon^t \geq 0$, since both δ_t and Ψ_t are positive or zero. Next we find an upper-bound for $\mathbb{E}\left[\Psi_{t+1}\right]$.

$$\begin{aligned}\mathbb{E}\left[\Psi_{t+1}\right] &= \frac{1}{n}(\frac{1}{n}\sum_{i=1}^{n} \|\nabla f_i(x_t) - \Gamma_{x^*}^{x_t}\left[\nabla f_i(x^*)\right]\|^2) \\ &+ (1-\frac{1}{n})(\frac{1}{n}\sum_{i=1}^{n} \|M^t[i] - \Gamma_{x^*}^{x_0}\left[\nabla f_i(x^*)\right]\|^2) \\ &= \frac{1}{n}(\frac{1}{n}\sum_{i=1}^{n} \|\nabla f_i(x_t) - \Gamma_{x^*}^{x_t}\left[\nabla f_i(x^*)\right]\|^2) + (1-\frac{1}{n})\Psi_t \\ &\leq \frac{L^2}{n}\delta_t + (1-\frac{1}{n})\Psi_t\end{aligned}$$

The inequality is due to the geodesic Lipschitz smoothness of f_i. Then, for some positive $\rho \leq 1$ we have the following inequality:

$$\begin{aligned}\mathbb{E}\left[\Upsilon^{t+1}\right] - (1-\rho)\Upsilon^t \leq (1 - \mu\eta + 2\zeta L^2\eta^2 - (1-\rho) + \frac{cL^2}{n})\delta_t \\ + (2\zeta\eta^2 - c(1-\rho) + c(1-\frac{1}{n}))\Psi_t. \qquad (1)\end{aligned}$$

In the right hand side of Inequality 1, δ_t and Ψ_t are positive by construction. If the coefficients of δ_t and Ψ_t in the right hand side of the Inequality 1 are negative, we would have $\mathbb{E}\left[\Upsilon^{t+1}\right] \leq (1-\rho)\Upsilon^t$. More precisely, we require

$$2\zeta\eta^2 - c(1-\rho) + c(1-\frac{1}{n}) \leq 0 \qquad (2)$$

$$1 - \mu\eta + 2\zeta L^2\eta^2 - (1-\rho) + \frac{cL^2}{n} \leq 0 \qquad (3)$$

To satisfy Inequality 2 we require $\rho \leq \frac{1}{n} - \frac{2\zeta\eta^2}{c}$. If we set $c = 2\alpha n\zeta\eta^2$ for some $\alpha > 1$, then $\rho \leq \frac{1}{n} - \frac{1}{\alpha n}$, which satisfies our requirement. If we replace the value of c in Inequality 3, we will get:

$$\begin{aligned}&\rho - \mu\eta + 2\zeta L^2\eta^2 + 2\alpha\zeta L^2\eta^2 \leq 0 \\ &\eta \in (\eta^- = \frac{2\mu - \sqrt{\mu^2 - 8\rho(1+\alpha)\zeta L^2}}{4(1+\alpha)\zeta L^2}, \eta^+ = \frac{2\mu + \sqrt{\mu^2 - 8\rho(1+\alpha)\zeta L^2}}{4(1+\alpha)\zeta L^2})\end{aligned}$$

To ensure the term under the square root is positive, we also need $\rho < \frac{\mu^2}{8(1+\alpha)\zeta L^2}$. Finally, if we set $\rho = \min\{\frac{\mu^2}{8(1+\alpha)\zeta L^2}, \frac{1}{n} - \frac{1}{\alpha n}\}$ and $\eta = \eta^+$, then we have:

$$\mathbb{E}\left[\Upsilon^{t+1}\right] \leq (1-\rho)^{t+1}\Upsilon^0,$$

where Υ^0 is a scalar. Since $\Psi_t > 0$ and $\mathbb{E}\left[\delta_{t+1}\right] \leq \mathbb{E}\left[\Upsilon^{t+1}\right]$, we get the required bound:

$$\mathbb{E}\left[\delta_{t+1}\right] \leq (1-\rho)^{t+1}\Upsilon^0.$$

Corollary 1. *Let* $\beta = \frac{n\mu^2}{8\zeta L^2}$*, and* $\bar{\alpha} = \beta + \sqrt{\frac{\beta^2}{4} + 1} > 1$*. If we set* $\alpha = \bar{\alpha}$ *then we will have* $\rho = \frac{\mu^2}{8(1+\bar{\alpha})\zeta L^2} = \frac{1}{n} - \frac{1}{\bar{\alpha} n}$*. Furthermore, to reach an* ϵ *accuracy,* i.e., $\mathbb{E}\left[\mathrm{d}^2(x_T, x^*)\right] < \epsilon$*, we require that the total number of MASAGA (Alg. 2) iteration* T *satisfy the following inequality:*

$$T \geq (\frac{8(1+\bar{\alpha})\zeta L^2}{\mu^2}) \log(\frac{1}{\epsilon}). \tag{4}$$

Note that this bound is similar to the bound of Zhang *et al.* [37]. To make it clear, notice that $\bar{\alpha} \leq 2\beta + 1$. Therefore, if we plug this upper-bound into Inequality 4 we get

$$T = \mathcal{O}(\frac{(2\beta+2)\zeta L^2}{\mu^2}) \log(\frac{1}{\epsilon}) = \mathcal{O}(\frac{n\mu^2}{8\zeta L^2}\frac{\zeta L^2}{\mu^2} + \frac{\zeta L^2}{\mu^2}) \log(\frac{1}{\epsilon}) = \mathcal{O}(n + \frac{\zeta L^2}{\mu^2}) \log(\frac{1}{\epsilon}).$$

The $\frac{L^2}{\mu^2}$ term in the above bound is the squared condition number that could be prohibitively large in machine learning applications. In contrast, the original SAGA and SVRG algorithms only depend on $\frac{L}{\mu}$ on convex function within linear spaces. In the next section, we improve upon this bound through non-uniform sampling techniques.

3.2 MASAGA with Non-uniform Sampling

Using non-uniform sampling for stochastic optimization in Euclidean spaces could help stochastic optimization methods achieve a faster convergence rate [31,21,9]. In this section, we assume that each f_i has its own geodesically L_i-Lipschitz smoothness as opposed to a single geodesic Lipschitz smoothness $L = \max\{L_i\}$. Now, instead of uniformly sampling f_i, we sample f_i with probability $\frac{L_i}{n\bar{L}}$, where $\bar{L} = \frac{1}{n}\sum_{i=1}^{n} L_i$. In machine learning applications, we typically have $\bar{L} \ll L$. Using this non-uniform sampling scheme, the iteration update is set to

$$x_{t+1} = \mathrm{Exp}_{x_t}(-\eta(\frac{\bar{L}}{L_{i_t}}\nu_t)),$$

which keeps the search direction unbiased, *i.e.*, $\mathbb{E}\left[\frac{\bar{L}}{L_{i_t}}\nu_t\right] = \nabla f(x_t)$. The following theorem shows the convergence of the new method.

Theorem 2. *If f_i is geodesically L_i-smooth and f is geodesically μ-strongly convex over the manifold $\mathcal{M}$, the MASAGA algorithm with the defined non-uniform sampling scheme and the constant step size $\eta = \frac{2\mu+\sqrt{\mu^2-8\rho(\bar{L}+\alpha L)\frac{\zeta}{\gamma}\bar{L}}}{4(\bar{L}+\alpha L)\frac{\zeta}{\gamma}\bar{L}}$ converges linearly as follows:*

$$\mathbb{E}\left[\mathrm{d}^2(x_t, x^*)\right] \leq (1-\rho)^t \Upsilon^0,$$

where $\rho = \min\{\frac{\gamma\mu^2}{8(1+\alpha)\zeta L\bar{L}}, \frac{\gamma}{n} - \frac{\gamma}{\alpha n}\}$, $\gamma = \frac{\min\{L_i\}}{\bar{L}}$, $L = \max\{L_i\}$, $\bar{L} = \frac{1}{n}\sum_{i=1}^n L_i$, and $\alpha > 1$ is a constant, and $\Upsilon^0 = \frac{2\alpha\zeta\eta^2}{\gamma}\sum_{i=1}^n \frac{\bar{L}}{L_i}\|M^0[i] - \Gamma_{x^*}^{x_0}\left[\nabla f_i(x^*)\right]\|^2 + \mathrm{d}^2(x_0, x^*)$ are positive scalars.

Proof of the above theorem could be found in the supplementary material.

Corollary 2. *Let $\beta = \frac{n\mu^2}{8\zeta L\bar{L}}$, and $\bar{\alpha} = \beta + \sqrt{\frac{\beta^2}{4}+1} > 1$. If we set $\alpha = \bar{\alpha}$ then we have $\rho = \frac{\gamma\mu^2}{8(1+\bar{\alpha})\zeta L\bar{L}} = \frac{\gamma}{n} - \frac{\gamma}{\bar{\alpha}n}$. Now, to reach an ϵ accuracy,* i.e., $\mathbb{E}\left[\mathrm{d}^2(x_T, x^*)\right] < \epsilon$, *we require:*

$$T = \mathcal{O}(n + \frac{\zeta L\bar{L}}{\gamma\mu^2})\log(\frac{1}{\epsilon}), \tag{5}$$

where T is the number of the necessary iterations.

Observe that the number of iterations T in Equality 5 depends on $\bar{L}L$ instead of L^2. When $\bar{L} \ll L$, the difference could be significant. Thus, MASAGA with non-uniform sampling could achieve an ϵ accuracy faster than MASAGA with uniform sampling.

Similarly we can use the same sampling scheme for the RSVRG algorithm [37] and improve its convergence. Specifically, if we change the update rule of Algorithm 1 of Zhang *et al.* [37] to

$$x_{t+1}^{s+1} = \mathrm{Exp}_{x_t^{s+1}} - \eta(\frac{\bar{L}}{L_{i_t}}\nu_t^{s+1}),$$

then Theorem 1 and Corollary 1 of Zhang *et al.* [37] will change to the following ones.

Theorem 3. ***[Theorem 1 of [37] with non-uniform sampling]*** *If we use non-uniform sampling in Algorithm 1 of RSVRG [37] and run it with the option I as described in the work, and let*

$$\alpha = \frac{3\zeta\eta\bar{L}^2}{\mu - 2\zeta\eta\bar{L}^2} + \frac{(1+4\zeta\eta^2 - 2\eta\mu)^m(\mu - 5\zeta\eta\bar{L}^2)}{\mu - 2\zeta\eta\bar{L}^2} < 1,$$

where m is the number of the inner loop iterations, then through S iterations of the outer loop, we have

$$\mathbb{E}\left[\mathrm{d}^2(\tilde{x}^S, x^*)\right] \leq (\alpha)^S \mathrm{d}^2(\tilde{x}^0, x^*)$$

The above theorem can be proved through a simple modification to the proof of Theorem 1 in RSVRG [37].

Corollary 3. ***[Corollary 1 of [37] with non-uniform sampling]*** *With non-uniform sampling in Algorithm 1 of RSVRG, after* $\mathcal{O}(n + \frac{\zeta \bar{L}^2}{\gamma \mu^2})\log(\frac{1}{\epsilon})$ *IFO calls, the output* x_a *satisfies*

$$\mathbb{E}\left[f(x_a) - f(x^*)\right] \leq \epsilon.$$

Note that through the non-uniform sampling scheme we improved the RSVRG [37] convergence by replacing the L^2 term with a smaller $\bar{L}^2$ term.

4 Experiments: Computing the leading eigenvector

Computing the leading eigenvector is important in many real-world applications. It is widely used in social networks, computer networks, and metabolic networks for community detection and characterization [24]. It can be used to extract a feature that "best" represents the dataset [8] to aid in tasks such as classification, regression, and background subtraction. Furthermore, it is used in the PageRank algorithms which requires computing the principal eigenvector of the matrix describing the hyperlinks in the web [13]. These datasets can be huge (the web has more than three billion pages [13]). Therefore, speeding up the leading eigenvector computation will have a significant impact on many applications.

We evaluate the convergence of MASAGA on the problem of computing the leading eigenvalue on several datasets. The problem is written as follows:

$$\min_{\{x \mid x^\top x = 1\}} f(x) = -\frac{1}{n} x^\top \left(\sum_{i=1}^{n} z_i z_i^\top \right) x, \tag{6}$$

which is a non-convex objective in the Euclidean space $\mathbb{R}^d$, but a (strongly-)convex objective over the Riemannian manifold. Therefore, MASAGA can achieve a linear convergence rate on this problem. We apply our algorithm on the following datasets:

- **Synthetic.** We generate Z as a 1000×100 matrix where each entry is sampled uniformly from $(0, 1)$. To diversify the Lipschitz constants of the individual z_i's, we multiply each z_i with an integer obtained uniformly between 1 and 100.
- **MNIST [17].** We randomly pick $10,000$ examples corresponding to digits 0-9 resulting in a matrix $Z \in \mathbb{R}^{10,000 \times 784}$.
- **Ocean.** We use the ocean video sequence data found in the UCSD background subtraction dataset [18]. It consists of 176 frames, each resized to a 94×58 image.

We compare MASAGA against RSGD [37] and RSVRG [4]. For solving geodesically smooth convex functions on the Riemannian manifold, RSGD and RSVRG achieve sublinear and linear convergence rates respectively. Since the manifold for Eq. 6 is that of a sphere, we have the following functions:

$$\begin{aligned} &P_X(H) = H - \text{trace}(X^\top H)X, && \nabla_r f(X) = P_X(\nabla f(X)), \\ &\text{Exp}_X(U) = \cos(||U||)X + \frac{\sin(||U||)}{||U||}U, && \Gamma_y^x(U) = P_y(U), \end{aligned} \tag{7}$$

where P corresponds to the tangent space projection function, $\nabla_r f$ the Riemannian gradient function, Exp the exponential map function, and Γ the transport function. We evaluate the progress of our algorithms at each epoch t by computing the relative error between the objective value and the optimum as $\frac{f(x^t)-f^*}{|f^*|}$. We have made the code available at `https://github.com/IssamLaradji/MASAGA`.

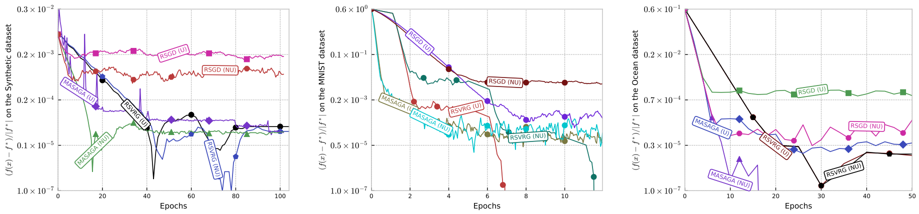
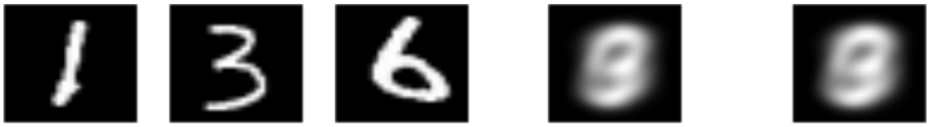


Fig. 1: Comparison of MASAGA (ours), RSVRG, and RSGD for computing the leading eigenvector. The suffix (U) represents uniform sampling and (NU) the non-uniform sampling variant.

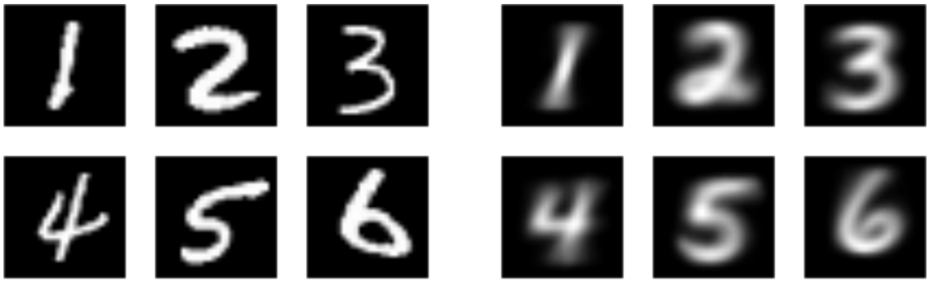


(a) Example MNIST images (b) Exact solution (c) MASAGA

Fig. 2: The obtained leading eigenvectors of all MNIST digits.

For each algorithm, a grid-search over the learning rates $\{10^{-1}, 10^{-2}, ..., 10^{-9}\}$ is performed and plot the results of the algorithm with the best performance in Figure 1. This plot shows that MASAGA is consistently faster than RSGD and RSVRG in the first few epochs. While it is expected that MASAGA beats RSGD since it has a better convergence rate, the reason MASAGA can outperform RSVRG is that RSVRG needs to occasionally re-compute the full gradient. Further, at each iteration MASAGA requires a single gradient evaluation instead of the two evaluations required by RSVRG. We see in Figure 1 that non-uniform (NU) sampling often leads to faster progress than uniform (U) sampling, which is consistent with the theoretical analysis. In the NU sampling case, we sample a vector z_i based on its Lipschitz constant $L_i = ||z_i||^2$. Note that for problems where L_i is not known or costly to compute, we can estimate it by using Algorithm 2 of Schmidt *et al.* [31].

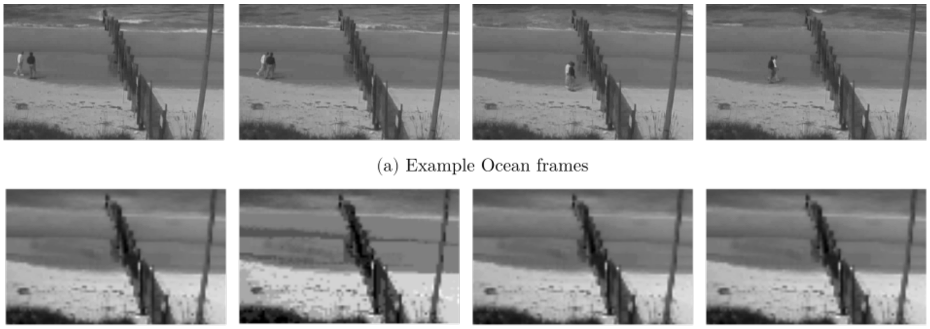
Figures 2 and 3 show the leading eigenvectors obtained for the MNIST dataset. We run MASAGA on $10{,}000$ images of the MNIST dataset and plot its solution in Figure 2. We see that the exact solution is similar to the solution obtained by MASAGA, which



(a) MNIST digits

(b) MASAGA for digits 1-6

Fig. 3: The obtained leading eigenvectors of the MNIST digits 1-6.



(a) Example Ocean frames

(b) Exact solution

(c) RSGD

(d) MASAGA

(e) RSVRG

Fig. 4: The obtained leading eigenvectors of the ocean dataset after 20 iterations.

represent the most common strokes among the MNIST digits. Furthermore, we ran MASAGA on 500 images for digits 1-6 independently and plot its solution for each class in Figure 3. Since most digits of the same class have similar shapes and are fairly centered, it is expected that the leading eigenvector would be similar to one of the digits in the dataset.

Figure 4 shows qualitative results comparing MASAGA, RSVRG, and RSGD. We run each algorithm for 20 iterations and plot the results. MASAGA's and RSVRG's results are visually similar to the exact solution. However, the RSGD result is visually different than the exact solution (the difference is in the center-left of the two images).

5 Conclusion

We introduced MASAGA which is a stochastic variance-reduced optimization algorithm for Riemannian manifolds. We analyzed the algorithm and showed that it converges linearly when the objective function is geodesically Lipschitz-smooth and strongly convex. We also showed that using non-uniform sampling improves the convergence speed of both MASAGA and RSVRG algorithms. Finally, we evaluated our method on a synthetic dataset and two real datasets where we empirically observed linear convergence. The empirical results show that MASAGA outperforms RSGD and is faster than RSVRG in the early iterations. For future work, we plan to extend MASAGA by deriving convergence rates for the non-convex case of geodesic objective functions. We also plan to explore accelerated variance-reduction methods and block coordinate descent based methods [26] for Riemannian optimization. Another potential future work of interest is a study of relationships between the condition number of a function within the Euclidean space and its corresponding condition number within a Riemannian manifold, and the effects of sectional curvature on it.

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